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LEVERED AND UNLEVERED BETA

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Pablo Fernández*

Abstract

We claim that in a world without leverage cost the relationship between the levered beta (β_L) and the unlevered beta (β_u) of a company depends upon the financing strategy. For a company that maintains a fixed book-value leverage ratio, the relationship is Fernández (2004): $\beta_L = \beta_u + (\beta_u - \beta_d) D (1 - T) / E$. For a company that maintains a fixed market-value leverage ratio, the relationship is Miles and Ezzell (1980): $\beta_L = \beta_u + (D / E) (\beta_u - \beta_d) [1 - T K_d / (1 + K_d)]$. For a company with a preset debt in every period, the relationship is Modigliani and Miller (1963): $\beta_L = \beta_u + [\beta_u - \beta_d] (D - VTS) / E$, where the Value of Tax Shields (VTS) is the present value of the future tax shields discounted at the cost of debt.

We also analyze alternative valuation theories proposed in the literature to estimate the relationship between the levered beta and the unlevered beta (Harris and Pringle (1985), Damodaran (1994), Myers (1974), and practitioners) and prove that all provide inconsistent results.

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LEVERED AND UNLEVERED BETA

This paper provides guidelines to evaluate the appropriateness of various relationships between the levered beta and the unlevered beta. We develop valuation formulae for a company that maintains a fixed book-value leverage ratio and claim that this is more realistic than to assume, as Miles and Ezzell (1980) do, a fixed market-value leverage ratio.

We prove that the relationship between the levered beta (β_L), the unlevered beta (β_u) and the beta of the debt (β_d) in a world with no leverage cost for a company that maintains a fixed book-value leverage ratio is:

$$[18] \beta_L = \beta_u + (\beta_u - \beta_d) D (1 - T) / E$$

In order to reach this result, we first prove that the value of tax shields (VTS) in a world with no leverage cost, for a constant growing company that maintains a fixed book-value leverage ratio, is the present value of the debt (D) times the tax rate (T) times the required return to the unlevered equity (K_u), discounted at the unlevered cost of equity (K_u):

$$[12] VTS = D T K_u / (K_u - g)$$

Please note that this does *not* mean that the appropriate discount for tax shields is the unlevered cost of equity. We discount $D T K_u$, which is higher than the tax shield. As shown in Fernández (2004), equation [12] is the difference of two present values.

The paper is organized as follows. In Section 1, we derive the relationship between the levered beta and the unlevered beta for growing perpetuities that maintain a fixed book-value leverage ratio in a world without leverage costs. This relationship is equation [18]. In Section 2, we review the financial literature on the relationship between the levered beta and the unlevered beta.

In Section 3 we analyze the seven theories for perpetuities. We prove that several theories provide inconsistent results: Harris-Pringle (1985), Miles-Ezzell (1980) Modigliani-Miller (1963), Myers (1974), and Practitioners.

Our conclusions are in Section 4. Appendix 1 contains a list of symbols and abbreviations used in the paper, and Appendix 2, the main valuation formulas according to the seven valuation theories that we analyze.

1. Relationship between the levered beta and the unlevered beta for growing perpetuities that maintain a fixed book-value leverage ratio in a world without leverage costs

The formula for the adjusted present value [1] indicates that the value of the debt today (D) plus that of the equity (E) of the levered company is equal to the value of the unlevered company (Vu) plus the value of tax shields due to interest payments (VTS).

$$[1] E + D = Vu + VTS$$

It is useful to get the relationship between the required return to equity (Ke), the required return to unlevered equity (Ku), the required return to debt (Kd), E, D, VTS and g (growth) for growing perpetuities. $E\{\cdot\}$ is the expected value operator. As $Vu = E\{FCF\} / (Ku - g)$, we can rewrite equation [1] as

$$[2] E + D = Vu = E\{FCF\} / (Ku - g) + VTS$$

In a growing perpetuity, the relationship between the expected equity cash flow ($E\{ECF\}$) and the expected free cash flow ($E\{FCF\}$) is

$$[3] E\{FCF\} = E\{ECF\} + D Kd (1 - T) - g D$$

By substituting [3] in [2], we get:

$$[4] E + D = [ECF + D Kd (1 - T) - g D] / (Ku - g) + VTS$$

As the relationship between the equity cash flow and the equity value is $ECF = E (Ke - g)$ we may rewrite [4] as:

$$[5] E + D = [E (Ke - g) + D Kd (1 - T) - g D] / (Ku - g) + VTS$$

Multiplying both sides of equation [5] by (Ku - g) we get:

$$[6] (E + D) (Ku - g) = [E (Ke - g) + D Kd (1 - T) - g D] + VTS (Ku - g)$$

Eliminating $-g (E + D)$ on both sides of equation [6]:

$$[7] (E + D) Ku = [E Ke + D Kd (1 - T)] + VTS (Ku - g)$$

Equation [7] may be rewritten as:

$$[8] D [Ku - Kd (1 - T)] - E (Ke - Ku) = VTS (Ku - g)$$

Fernández (2006) proves in his equation (12) that the Value of tax shields is

$$[9] VTS_0 = T \sum_1^{\infty} PV_0 [Interest_t] = T \cdot D_0 + T \sum_1^{\infty} PV_0 [\Delta D_t]$$

Equation [9], valid for companies with any pattern of growth, shows that the value of tax shields depends *only* upon the nature of the stochastic process of the net increase of debt. The value today of the expected increases of debt depends on the financing strategy.

If the company has a preset amount of debt, the future increases of debt (ΔD_t) are known with certainty today and Modigliani-Miller (1963) applies: the appropriate discount rate for ΔD_t is R_F , the risk-free rate. If the debt is expected to increase at a constant rate g, then $PV_0 [\Delta D_t] = \Delta D_0$

$(1+g)^t/(1+R_F)^t$ and equation [9] is the sum of a geometric progression with growth rate $(1+g)/(1+R_F)$. Then:

$$[10] \quad VTS_0 = TD_0 + T \frac{gD_0}{R_F - g} = \frac{D_0 R_F T}{(R_F - g)}$$

Fieten et al. (2005) argue that the Modigliani-Miller formula may be applied to all situations. However, it is valid only when the company has a preset amount of debt.

Miles and Ezzell (1980) and Arzac and Glostén (2005) assume that debt is proportional to equity market value in every period ($D_t = L \cdot S_t$). If $D_t = L \cdot S_t$, the appropriate discount rate for S_t is equal to the required return to the value of debt. As VTS is proportional to D , following equation [1], D_t , S_t , Vu_t and VTS_t have the same risk and the appropriate discount rate for all of them is Ku , because the appropriate discount rate for Vu_t is Ku . Then, the value today of the increase of debt in period 1 is:

$$PV_0[\Delta D_1] = \frac{D_0(1+g)}{1+Ku} - \frac{D_0}{1+R_F}$$

The present value of the expected increase of debt in period t (as D_{t-1} is known in period $t-1$) is:

$$PV_0[\Delta D_t] = \frac{D_0(1+g)^t}{(1+Ku)^t} - \frac{D_0(1+g)^{t-1}}{(1+R_F)(1+Ku)^{t-1}}$$

The sum of all the present values of the expected increases of debt is a geometric progression with growth rate $= (1+g)/(1+Ku)$. The sum is:

$$\sum_{t=1}^{\infty} PV_0[\Delta D_t] = \frac{D_0}{(Ku - g)} \left(g - \frac{Ku - R_F}{1 + R_F} \right)$$

Substituting the last equation in [9], we get the well known Miles-Ezzell formula:

$$[11] \quad VTS_0 = \frac{D_0 R_F T (1 + Ku)}{(Ku - g) (1 + R_F)}$$

To assume $D_t = L \cdot S_t$ is not a good description of the debt policy of any company because if a company has only two possible states of nature in the following period, under the worst state (low share price) the company will have to raise new equity and repay debt, and this is not the moment companies prefer to raise equity. Under the good state, the company will have to take on a lot of debt and pay big dividends.

The Miles-Ezzell setup is equivalent to assuming that the increase of debt is proportional to the increase of the free cash flow in every period.

Fernández (2006) shows that for a company with a fixed book-value leverage ratio, the increase of debt is proportional to the increases of net assets, and the risk of the increases of debt is equal to the risk of the increases of assets. If Ku is the appropriate discount rate for the expected increases of the book value of assets, then:

$$\sum_{t=1}^{\infty} PV_0[\Delta D_t] = \frac{gD_0}{Ku - g} \quad (12)$$

Substituting the last equation in [9], we get:

$$[12] VTS = D T K_u / (K_u - g)$$

Substituting equation [12] in [8], we get:

$$[13] K_e = K_u + (D / E) (1 - T) (K_u - K_d)$$

The formulas relating the betas to the required returns are:

$$[14] K_e = R_F + \beta_L P_M$$

$$[15] K_u = R_F + \beta_u P_M$$

$$[16] K_d = R_F + \beta_d P_M$$

R_F is the risk-free rate and P_M is the market risk premium. Substituting [14], [15] and [16] in [13], we get:

$$[17] R_F + \beta_L P_M = R_F + \beta_u P_M + (R_F + \beta_u P_M - R_F - \beta_d P_M) D (1 - T) / E$$

Then, the relationship between the beta of the levered equity (β_L), the beta of the unlevered equity (β_u) and the beta of debt (β_d) for a company with a fixed book-value leverage ratio in a world *without leverage costs* is:

$$[18] \beta_L = \beta_u + (\beta_u - \beta_d) D (1 - T) / E$$

Equation [12], applied to the general case, is (see Fernández (2004)):

$$[19] VTS = PV[K_u; D T K_u]$$

2. Literature review

There is a considerable body of literature on the discounted cash flow valuation of firms. We will now discuss the most salient papers, concentrating particularly on those that propose different expressions for the relationship between levered beta and unlevered beta.

Modigliani and Miller (1958) studied the effect of leverage on the firm's value. In the presence of taxes and for the case of a perpetuity, they calculate the value of tax shields by discounting the present value of the tax savings due to interest payments of a risk-free debt ($T D R_F$) at the risk-free rate (R_F). Their first proposition, with taxes, is transformed into Modigliani and Miller (1963, page 436, formula 3):

$$[21] E + D = V_u + PV[R_F; D T R_F] = V_u + D T$$

DT is the value of tax shields for a perpetuity. This result is the same as our equation [12] applied to perpetuities. But as will be proven later on, this result is correct only for perpetuities. Discounting the tax savings due to interest payments of a risk-free debt at the risk-free rate provides inconsistent results for growing companies. Modigliani and Miller's purpose was to illustrate the tax impact of debt on value. They never addressed the issue of the riskiness of the taxes and only treated perpetuities. Later on, it will be seen that if we relax the no-growth assumption, then new formulas are needed.

For a perpetuity, the relationship between levered beta and unlevered beta implied by [21] is [18]. But for a growing perpetuity, the value of tax shields for a growing perpetuity, according to Modigliani and Miller (1963), is:

$$[22] \text{ VTS} = D T R_F / (R_F - g)$$

Sick (1990) and Fieten et al. (2005) also recommend formula [22].

Substituting [22] in [8], we get:

$$D [K_u - K_d (1 - T)] - E (K_e - K_u) = D T R_F (K_u - g) / (R_F - g)$$

Then, the relationship between the levered and the unlevered required return to equity according to Modigliani and Miller (1963) is:

$$[23] K_e = K_u + (D / E) [K_u - K_d (1 - T) - T R_F (K_u - g) / (R_F - g)] = \\ = K_u + (D / E) [K_u - K_d (1 - T) - \text{VTS} (K_u - g) / D]$$

And the relationship between levered beta and unlevered beta is

$$[24] \beta_L = \beta_u + (D / E) [\beta_u - \beta_d + (T K_d / P_M) - \text{VTS} (K_u - g) / (D P_M)]$$

Myers (1974) introduced the APV (adjusted present value). According to this, the value of the levered firm is equal to the value of the firm with no debt (V_u) plus the present value of the tax saving due to the payment of interest. Myers proposes calculating the value of tax shields by discounting the tax savings at the cost of debt (K_d). The argument is that the risk of the tax saving arising from the use of debt is the same as the risk of the debt. Then, according to Myers (1974):

$$[25] \text{ VTS} = \text{PV} [K_d; D T K_d]$$

It is easy to deduce that the relationship between levered beta and unlevered beta implied by [25] for growing perpetuities is [26]:

$$[26] \beta_L = \beta_u + (D / E) (\beta_u - \beta_d) [1 - T K_d / (K_d - g)]$$

Luehrman (1997) recommends that companies be valued using the Adjusted Present Value and calculates the VTS in the same way as Myers. This theory provides inconsistent results for companies other than perpetuities, as will be shown later.

According to Miles and Ezzell (1980), Arzac and Glosten (2005), and Cooper and Nyborg (2006), the correct rate for discounting the tax saving due to debt ($K_d T D$) of a firm with a fixed debt target $[D/(D+E)]$ (market value) is K_d for the tax saving in the first year, and K_u for the tax saving in following years. The expression of K_e is their formula (22):

$$[27] K_e = K_u + D (K_u - K_d) [1 + K_d (1 - T)] / [(1 + K_d) E]$$

And the relationship between levered beta and unlevered beta implied by [27] is their formula (27) in Miles and Ezzell (1985):

$$[28] \beta_L = \beta_u + (D / E) (\beta_u - \beta_d) [1 - T K_d / (1 + K_d)]$$

Lewellen and Emery (1986) also claim that the most logically consistent method is Miles and Ezzell.

Harris and Pringle (1985) propose that the present value of tax shields should be calculated by discounting the tax saving due to the debt ($D K_d T$) at the required return to assets. Their argument is that the interest tax shields have the same systematic risk as the firm's underlying cash flows and, therefore, should be discounted at the required return to assets. Then, according to Harris and Pringle (1985), the value of tax shields is:

$$[29] \text{ VTS} = \text{PV} [K_u; D K_d T]$$

Substituting [29] for growing perpetuities in [8], we get:

$$[30] D [K_u - K_d (1 - T)] - E (K_e - K_u) = D K_d T$$

Then, the relationship between the levered and the unlevered required return to equity according to Harris and Pringle (1985) is:

$$[31] K_e = K_u + (D / E) (K_u - K_d)$$

And the relationship between levered beta and unlevered beta implied by [31] is:

$$[32] \beta_L = \beta_u + (D / E) (\beta_u - \beta_d)$$

Ruback (1995) reaches formulas that are identical to those of Harris-Pringle (1985). Kaplan and Ruback (1995) use the Compressed APV method and also calculate the VTS “discounting interest tax shields at the discount rate for an all-equity firm. This assumes that the interest tax shields have the same systematic risk as the firm's underlying cash flows”. But their equation (4) is equivalent to equation [18]. Kaplan and Ruback (1995) mix two theories: they use the Fernández theory to unlever the beta and the Harris-Pringle theory to calculate the value of tax shields. Tham and Vélez-Pareja (2001), following an arbitrage argument, also claim that the appropriate discount rate for tax shields is K_u , the required return to unlevered equity. Brealey and Myers (2000, page 555) also recommend [32] “for relevering beta”.

Taggart (1991) gives a good summary of valuation formulas with and without personal income tax. He proposes that Miles & Ezzell's (1980) formulas should be used when the company adjusts to its target debt ratio once a year and Harris & Pringle's (1985) formulas when the company adjusts to its target debt ratio continuously.

Damodaran (1994, pages 31 and 277) argues that if all the business risk is borne by the equity, then the formula relating the levered beta (β_L) to the asset beta (β_u) is:

$$[33] \beta_L = \beta_u + (D / E) \beta_u (1 - T).$$

It is important to note that formula [33] is exactly formula [18], assuming that $\beta_d = 0$. One interpretation of this assumption is (see page 31 of Damodaran, 1994) that “all of the firm's risk is borne by the stockholders (i.e., the beta of the debt is zero) and debt has a tax benefit to the firm”. But we think that, in general, it is difficult to justify that the debt has no risk and that the return on the debt is uncorrelated with the return on assets of the firm. Instead, we interpret formula [33] as an attempt to introduce some leverage cost in the valuation: for a given risk of the assets (β_u), by using formula [33] we obtain a higher β_L (and consequently a higher K_e and a lower equity value) than with formula [18]. Equation [33] appears in many finance books and is used by many consultants and investment bankers.

In some cases it may be not so outrageous to give debt a beta of 0. But if this is the case, then the required return to debt is the risk-free rate.

Damodaran also says (footnote on page 31) “if debt has market risk, the beta of equity can be written as

$$[34] \beta_L = \beta_u + (D / E) \beta_u (1 - T) - \beta_d (D / E).”$$

Comparing this expression with [18], we may conclude that [34] provides a lower value of β_L than [18].

Copeland, Koller and Murrin (2000, page 309) use formula [33], but in their Appendix A (page 482) they propose formula [32], that of Harris y Pringle (1985), to lever the beta. They also claim that “the finance literature does not provide a clear answer about which discount rate for the tax benefit of interest is theoretically correct.” And they conclude “we leave it to the reader’s judgment to decide which approach best fits his or her situation.” It is quite interesting to note that Copeland et al. (2000, page 483) only suggest Inselbag and Kaufold (1997) as additional reading on Adjusted Present Value.

Formula (4a) of Hamada (1972) is also equal to [33], although Hamada assumed that the value of tax shields is equal to T D.

Another way of calculating the levered beta with respect to the asset beta is the following:

$$[35] \beta_L = \beta_u (1 + D / E).$$

We will call this method the **Practitioners’ method** because it is often used by consultants and investment banks (one of the many places where it appears is Ruback, 1995, page 5). It is obvious that according to this formula, given the same value for β_u , a higher β_L (and a higher K_e and a lower equity value) is obtained than according to [18] and [33].

One should notice that formula [35] is equal to formula [33] eliminating the $(1 - T)$ term. We interpret formula [35] as an attempt to introduce still higher leverage cost in the valuation: for a given risk of the assets (β_u), by using formula [35] we obtain a higher β_L (and consequently a higher K_e and a lower equity value) than with formula [33].

It is important to note that Damodaran (1994) and Practitioners impose a cost of leverage, but they do so in an ad hoc way.

Inselbag and Kaufold (1997) argue that if the firm targets the dollar values of debt outstanding, the firm should be valued using the Myers (1974) formulae. However, if the firm targets a constant debt/value ratio, the firm should be valued using the Miles and Ezzell (1980) formulae.

3. Analysis of the seven theories for growing perpetuities

Table 1 reports the relationship between levered beta and unlevered beta of the seven theories for the case of growing perpetuities.

Table 1

Levered beta according to the seven theories

	Theories		Formula
1	Fernández	[18]	$\beta_L = \beta_u + (\beta_u - \beta_d) D (1 - T) / E$
2	Damodaran	[33]	$\beta_L = \beta_u + (D / E) \beta_u (1 - T)$.
3	Practitioners	[35]	$\beta_L = \beta_u (1 + D / E)$.
4	Harris-Pringle	[32]	$\beta_L = \beta_u + (D / E) (\beta_u - \beta_d)$
5	Myers	[26]	$\beta_L = \beta_u + (D / E) (\beta_u - \beta_d) [1 - T K_d / (K_d - g)]^*$
6	Miles-Ezzell	[28]	$\beta_L = \beta_u + (D / E) (\beta_u - \beta_d) [1 - T K_d / (1 + K_d)]$
7	Modigliani-Miller	[24]	$\beta_L = \beta_u + (D / E) [\beta_u - \beta_d + (T K_d / P_M) - VTS (K_u - g) / (D P_M)]^*$

* Valid only for growing perpetuities

From the relationships between β_L and β_u in Table 1, we may extract some consequences that affect the validity of the theories. These consequences are summarized in Table 2:

- The Fernández formula always provides us with $\beta_L > \beta_u$ because β_u is always higher than β_d .
- Myers provides us with the inconsistent result (for growing perpetuities) of β_L being lower than β_u if the value of tax shields is higher than the value of debt. This happens when $D T K_d / (K_d - g) > D$, that is, when the growth rate is higher than the after-tax cost of debt: $g > K_d (1 - T)$. Note that in this situation, as the value of tax shields is higher than the value of debt, the equity (E) is worth more than the unlevered equity (V_u). This result makes no economic sense.
- Modigliani-Miller provides us with the inconsistent result of β_L being lower than β_u if the value of tax shields is higher than $D [K_u - K_d (1 - T)] / (K_u - g)$. This happens when the leverage, the tax rate, the cost of debt or the market risk premium are high.

Table 2

Problems of Myers and Modigliani-Miller in a world with constant growth: The levered beta may be lower than the unlevered beta

	Levered beta < Unlevered beta
Myers	If $g > K_d (1 - T)$
Modigliani-Miller	If $g > R_F (1 - T) / [1 + T \beta_d / (\beta_u - \beta_d)]$

Now we use the seven formulae of Table 1 for a hypothetical company with constant annual growth of 4%. The company has \$30 million of debt at 8% and the equity (book value) is also \$30 million. Net fixed assets are also equal to working capital requirements. The expected net income for next year is \$5.46 million and the expected free cash flow is \$5.44 million. Depreciation will be \$6 million and expected investment in fixed assets is \$7.2 million. The

corporate tax rate is 40%, the risk-free rate is 6.5%, and the market risk premium is 5%. The unlevered beta is 0.7.

Table 3 offers us the sensitivity analysis of the seven theories for an example with constant growth. It may be seen that without growth, the Myers and Modigliani-Miller formulae equal the Fernández formula. Obviously, the levered beta (β_L) should be higher than the unlevered beta (β_u) because the equity cash flow is riskier than the free cash flow. But, with growth, $\beta_L < \beta_u$ according to Myers (for $g > 4.8\%$) and according to Modigliani-Miller (for $g > 3\%$).

Table 3

Sensitivity of the levered beta to the growth rate. $\beta_u = 0.7$. $T = 40\%$

Growth rate:	0.00%	1.00%	3.00%	4.00%	5.00%	6.00%	7.00%	7.50%
Modigliani-Miller	0.84400	0.80689	0.70000	0.61628	0.48776	0.24563	-0.55638	-3.68750
Myers	0.84400	0.82384	0.77125	0.73564	0.68974	0.62653	0.52707	0.44488
Fernández	0.84400	0.83787	0.82293	0.81368	0.80286	0.79000	0.77448	0.76545
Miles & Ezzell	0.94372	0.93404	0.91020	0.89528	0.87763	0.85642	0.83046	0.81516
Harris-Pringle	0.95210	0.94215	0.91762	0.90225	0.88405	0.86216	0.83534	0.81952
Damodaran	0.96638	0.95598	0.93029	0.91416	0.89505	0.87201	0.84373	0.82702
Practitioners	1.18724	1.17132	1.13109	1.10514	1.07367	1.03466	0.98507	0.95485

Table 4 offers us the sensitivity analysis of the levered beta to the tax rate for our example. It may be seen that in all situations both Damodaran and Practitioners provide us with a higher levered beta than the Fernández (2004) formula.

The Harris-Pringle (1985) and Miles-Ezzell (1980) formulae equal the Fernández formula when $T = 0$ (no taxes); but when $T > 0$, both provide a higher levered beta than the Fernández (2004) formula. We may conclude, therefore, that both Harris and Pringle (1985) and Miles and Ezzell (1980) provide inconsistent results. They are not appropriate for valuing companies without leverage cost because the Fernández (2004) formula is the right one.

According to Myers and Modigliani-Miller, β_L is lower than β_u when the tax rate is higher than 50% and 30%. Furthermore, according to Myers and Modigliani-Miller, β_L decreases when the tax rate increases. According to the Fernández (2004) formula, β_L increases when the tax rate increases.

Tabla 4

Sensitivity of the levered beta to the tax rate. $\beta_u = 0.7$, $g = 4\%$

Taxes	0.00%	20.00%	30.00%	40.00%	50.00%	60.00%	70.00%	75.00%
Myers	0.80093	0.77733	0.75983	0.73564	0.70000	0.64225	0.53256	0.43000
Modigliani-Miller	0.80093	0.72978	0.68038	0.61628	0.52977	0.40662	0.21729	0.07854
Harris-Pringle	0.80093	0.83466	0.86168	0.90225	0.97000	1.10602	1.51818	2.36154
Miles & Ezzell	0.80093	0.83245	0.85761	0.89528	0.95785	1.08222	1.44927	2.15714
Fernández	0.80093	0.80537	0.80878	0.81368	0.82135	0.83500	0.86615	0.90377
Damodaran	0.88853	0.89739	0.90425	0.91416	0.92979	0.95802	1.02446	1.10865
Practitioners	0.88853	0.95732	1.01474	1.10514	1.26842	1.65214	3.63023	307.90182

Table 5 offers us the sensitivity analysis of the levered beta to the tax rate for the no-growth company. It may be seen that for perpetuities, the levered beta does not depend on the tax rate according to Damodaran and the Fernández (2004) formula. However, according to Practitioners, Harris-Pringle and Miles-Ezzel, levered betas grow with tax rates. This result does not make much economic sense.

Table 5

Sensitivity of the levered beta to the tax rate. $\beta_u = 0.7$, $g = 0\%$

Taxes	0.00%	20.00%	40.00%	50.00%	60.00%	70.00%	75.00%
Fernández	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400
Myers	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400
Modigliani-Miller	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400	0.84400
Miles & Ezzell	0.84400	0.88034	0.94372	0.99714	1.08222	1.23895	1.38000
Harris-Pringle	0.84400	0.88330	0.95210	1.01034	1.10359	1.27692	1.43469
Damodaran	0.96638	0.96638	0.96638	0.96638	0.96638	0.96638	0.96638
Practitioners	0.96638	1.04445	1.18724	1.31463	1.53223	1.98834	2.47465

Table 6 offers us a sensitivity analysis of unlevering the beta for an example with constant growth and levered beta equal to one. It may be seen that the higher the debt-to-equity ratio, the wider the range of unlevered betas.

Table 6

Calculating the unlevered beta. $\beta_L = 1$, $R_F = 6.5\%$, $P_M = 5\%$, $T = 40\%$, $g = 3\%$, $K_d = 7.5\%$

Debt to equity (D/E)	20%	40%	50%	60%	100%	150%	200%
Fernández	0.914	0.845	0.815	0.788	0.700	0.621	0.564
Damodaran	0.893	0.806	0.769	0.735	0.625	0.526	0.455
Practitioners	0.833	0.714	0.667	0.625	0.500	0.400	0.333
Harris-Pringle	0.867	0.771	0.733	0.700	0.600	0.520	0.467
Myers	0.941	0.916	0.886	0.867	0.800	0.698	0.680
Modigliani-Miller	0.974	0.950	0.939	0.929	0.891	0.852	0.819
Miles & Ezzell	0.870	0.776	0.738	0.705	0.606	0.527	0.472

4. Conclusions

This paper provides clear, theoretically sound guidelines to evaluate the appropriateness of various relationships between the levered beta and the unlevered beta.

For constant growth companies, we claim that the relationship between the levered beta (β_L) and the unlevered beta (β_u) for a company that maintains a fixed book-value leverage ratio in a world with no leverage cost is Fernández (2004):

$$[18] \beta_L = \beta_u + (\beta_u - \beta_d) D (1 - T) / E.$$

We also compare that formula with those of Harris and Pringle (1985), Modigliani and Miller (1963), Damodaran (1994), Myers (1974), Miles and Ezzell (1980), and practitioners.

Formula [18] is a consequence of the value of tax shields for a company that maintains a fixed book-value leverage ratio in a world with no leverage cost:

$$[12] \text{ VTS} = D T K_u / (K_u - g)$$

In order to operationalize a valuation, very often one begins with assumptions of β_d and β_L , not with β_u . β_u has to be inferred from β_d and β_L . Which theories allow us to calculate β_u ? Without leverage costs, the most sensible relationship between the betas is equation [18].

Appendix 1

Symbols and Abbreviations

β_d = Beta of debt

β_L = Beta of levered equity

β_u = Beta of unlevered equity = Beta of assets

D = Value of debt

E = Value of equity

ECF = Equity cash flow

FCF = Free cash flow

g = Growth rate of the constant growth case

I = Interest paid = D K_d

K_u = Cost of unlevered equity (required return to unlevered equity)

K_e = Cost of levered equity (required return to levered equity)

K_d = Required return to debt = Cost of debt

LC = Leverage cost

P_M = Market risk premium = E (R_M - R_F)

PV = Present value

R_F = Risk-free rate

T = Corporate tax rate

VTS = Value of the tax shields

V_u = Value of shares in the unlevered company

WACC = Weighted average cost of capital

Appendix 2

Main valuation formulas

Market value of the debt = Nominal value

	Fernández (2004)	Damodaran (1994)	Practitioners
K_e	[13] $K_e = K_u + \frac{D(1-T)}{E}(K_u - K_d)$	$K_e = K_u + \frac{D(1-T)}{E}(K_u - R_F)$	$K_e = K_u + \frac{D}{E}(K_u - R_F)$
β_L	[18] $\beta_L = \beta_u + \frac{D(1-T)}{E}(\beta_u - \beta_d)$	[33] $\beta_L = \beta_u + \frac{D(1-T)}{E}\beta_u$	[35] $\beta_L = \beta_u + \frac{D}{E}\beta_u$
β_u	$\beta_u = \frac{E\beta_L + D(1-T)\beta_d}{E + D(1-T)}$	$\beta_u = \frac{E\beta_L}{E + D(1-T)}$	$\beta_u = \frac{E\beta_L}{E + D}$
WACC	$K_u \left(1 - \frac{DT}{E + D}\right)$	$K_u \left(1 - \frac{DT}{E + D}\right) + D \frac{(K_d - R_F)(1-T)}{(E + D)}$	$K_u - D \frac{R_F - K_d(1-T)}{(E + D)}$
VTS	[19] $PV[K_u; DTK_u]$	$PV[K_u; DTK_u - D(K_d - R_F)(1-T)]$	$PV[K_u; DTK_d - D(K_d - R_F)]$

	Harris-Pringle (1985) Ruback (1995)	Myers (1974)	Miles-Ezzell (1980)
K_e	[31] $K_e = K_u + \frac{D}{E}(K_u - K_d)$	$K_e = K_u + \frac{D - DVTS}{E}(K_u - K_d)$	[27] $K_e = K_u + \frac{D}{E}(K_u - K_d) \left[1 - \frac{T K_d}{1 + K_d}\right]$
β_L	[32] $\beta_L = \beta_u + \frac{D}{E}(\beta_u - \beta_d)$	[26] $\beta_L = \beta_u + \frac{D - DVTS}{E}(\beta_u - \beta_d)$	[28] $\beta_L = \beta_u + \frac{D}{E}(\beta_u - \beta_d) \left[1 - \frac{T K_d}{1 + K_d}\right]$
β_u	$\beta_u = \frac{E\beta_L + D\beta_d}{E + D}$	$\beta_u = \frac{E\beta_L + (D - VTS)\beta_d}{V_u}$	$\beta_u = \frac{E\beta_L + D\beta_d [1 - T K_d / (1 + K_d)]}{E + D [1 - T K_d / (1 + K_d)]}$
WACC	$K_u - D \frac{K_d T}{(E + D)}$	$K_u - \frac{DVTS(K_u - K_d) + D K_d T}{(E + D)}$	$K_u - D \frac{K_d T}{(E + D)} \frac{1 + K_u}{1 + K_d}$
VTS	[29] $PV[K_u; D T K_d]$	[25] $PV[K_d; D T K_d]$	$PV[K_u; D T K_d] (1 + K_u) / (1 + K_d)$

	Modigliani-Miller
K_e	[23] $K_e = K_u + \frac{D}{E} [K_u - K_d(1-T) - (K_u - g) \frac{VTS}{D}]^*$
β_L	[24] $\beta_L = \beta_u + \frac{D}{E} [\beta_u - \beta_d + \frac{TK_d}{P_M} - \frac{VTS(K_u - g)}{D P_M}]^*$
β_u	$\beta_u = \frac{E\beta_L + D\beta_d - [DTK_d - VTS(K_u - g)]/P_M}{E + D}^*$
WACC	$\frac{D K_u - (K_u - g) VTS}{(E + D)}^*$
VTS	[22] $PV[R_F; D T R_F]$

* Valid only for growing perpetuities.

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